How Cumbersome is a Tenth Order Polynomial?: The Case of Gravitational Triple Lens Equation

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ABSTRACT

Three point mass gravitational lens equation is a two-dimensional vector equation that can be embedded in a tenth order analytic polynomial equation of one complex variable, and we can solve the one variable equation on the source trajectories using recipies for Fortran or C (portable for C++ or C_{jj}) in Numerical Recipes, or using packages such as Mathemetica, Matlab, etc. This ready solvability renders fitting microlensing light curves including triple lenses a normal process, and such was done in a circumbinary planet fit for MACHO-97-BLG-41. Subsequently, there was a claim that converting the triple lens equation into the analytic equation was rather cumbersome, and the impressionable judgement has caused an effect of mysterious impedance around the perfectly tractable lens equation. There are judgements. Then, there is nature. We looked up for one of the quantities of highest precision measurements: electron g-factor correction $a_e \equiv g/2 - 1$. The current best experimental values of a_e agree to eight significant digits with the theoretical value, and the theoretical calculation involves more than one thousand Feynman diagrams – many orders of magnitude messier than the triple lens equation coefficients. We seem to have only choice to be compliant to nature and its appetite for elegant mess and precision numerics. In fact, the triple lens equation coefficients take up less than a page to write out and are presented here for users' convenience.

Subject headings: gravitational lensing:

1. Gravitational Triple Lens Equation

The lens equation of a set of three gravitationally bound point masses is written with three (real) relative mass parameters, ϵ_1, ϵ_2, and ϵ_3, and three complex parameters for their 2-dimensional positions. If z is the position of an image and ω is the position of its source, the lens equation reads as follows.

$$\omega = z - \frac{\epsilon_1}{\bar{z} - \bar{x}_1} - \frac{\epsilon_2}{\bar{z} - \bar{x}_2} - \frac{\epsilon_3}{\bar{z} - \bar{x}_3} \equiv z - f(\bar{z}; \bar{x}_j) \tag{1}$$

The mass parameters are subject to a constraint $\epsilon_1 + \epsilon_2 + \epsilon_3 = 1$ where $1 = M$ is the total mass, and the position variables include coordinate degrees of freedom: two degrees of freedom for translation and one degree of freedom for roation in the two-dimensional lens plane. We choose a coordinate system such that the position of a mass, x_1, and the center of mass of the other two elements, x_4, define the lens axis along the real axis of the complex plane.

$$(\epsilon_2 + \epsilon_3)\ x_4 = \epsilon_2 x_2 + \epsilon_3 x_3 \tag{2}$$

Then, x_1 and x_4 are real, and the triple lens system is completely specified by a set of five parameters.

$$\ell = |x_1 - x_4|\ ; \quad \ell' = |x_2 - x_3|\ ; \quad \gamma = [0, \pi]\ ; \quad \epsilon_2 = (0, 1)\ ; \quad \epsilon_3 = (0, 1) \tag{3}$$

The angle γ is the angle between $(x_2 - x_3)$ and the lens axis. In order to find the images of a given source, we need to solve the lens equation.

When the lensing system and a source star are far apart, there are four images of the source star, which we can verify easily for a source at $\omega = \infty$. From the lens equation (1), we find that the images of the source at $\omega = \infty$ are at the position of the source, $z = \infty$, and at the three lens positions, $z = x_1, x_2$, and x_3. The magnifications of the images at the lens positions are 0, and they are hardly images in any practical sense because there are no photon fluxes related to the image positions. In other words, there is only one physical image for a source at ∞ – namely, the image of the unmagnified source star. However, if we move the source toward the lens system, the images at the lens positions move away from the lens positions, and they do have non-vanishing photon fluxes. Thus, we freely speak of image positions with zero magnification as the continuity limit so that we don't have to cut out the lens positions from the image space. In practice, ∞ is a large distance limit, which may be no more than a few hundred au in any physical relevance of Galactic lensing.

If we consider a source trajectory away from the caustic regions, the images form four smooth curves anchored at the *base points* of $z = x_1, x_2, x_3$, and ∞ which are the image positions of the source at $\omega = \infty$. The complex lens plane can be considered a large two sphere with one point at ∞, and then, the four smooth disjoint image curves are loops with fixed points at the *base points*. The image curves can be calculated esaily from the *base points* using Newtonian transportation method (the algorithm can be found in Numerical Recipes). However, the physical interest of lensing lies in caustic regions where the lensing signals are most obvious, and this makes the Newtonian method useless in microlensing business. The large lensing signals arise where the inverse Jacobian determinant of the lens equation is large, and the large Jacobian determinant makes Newtonian transportation method unstable and unusable. The method of infinitesimal corrections based on derivative values is valid where the Jacobian determinant is practically finite and the mapping is non-degenerate.

Laguerre's method (see Numerical Recipes) is based on complex analysis and uses first and second derivatives of the polynomial to help converge to the solutions from initial values which one is free to guess.

Another complication of the caustic region is that the size (which may be $1-10\mu$as) of the source star can not be ignored when the source star crosses a caustic curve. In fact, the luminosity profile of the source star can manifestly affect the shape of the light curve, and the luminosity profile dependence of the light curve shape during a line caustic crossing lasts ~ 3 stellar radius crossing times (Rhie and Bennett 1999). In order to incorporate the finite size effects, one can *locally* pixelize the image plane (z-plane) in the neighborhood of the critical curve near the images and count up the image pixels that are mapped into the source disk (Bennett and Rhie 1996). The luminosity profile of the source can be easily incorporated by weighting each image pixel by the luminosity shape function value at the corresponding source position. The pixel size can be adjusted for a desired resolution. In the ray shooting method which has been designed mainly to handle a system of a large number of lensing elements solving whose lens equation is impractical if not impossible, the entire lens plane is pre-pixelized and each pixel in the image plane is tested whether the center of the pixel is mapped into the area defined by the source disk in the source plane. The ray shooting method is inefficient for fitting light curves of low-multiplicity point mass lenses as microlensing planet systems.

The astro-ph version of Gaudi, Naber, and Sackett (1998) conveys an impression that solving the lens equation by finding the roots of the 10-th order polynomial analytic equation is slower than using the ray shooting method for triple lens light curve fitting. However, Gaudi (private communication) recently informed us that the comment on the calculation speed was a comparison between binary lens and triple lens but not a comparison between root finding method and ray shooting method for the triple lens equation.

2. The Tenth Order Polynomial Equation

The lens equation is an explicit function from an image position z to its source position ω, and $\omega(z,\bar{z})$ is a genuine real function, namely a function of both z and $\bar{z}$. However, the z-depdence is linear, and this simplicity is behind the analyticity of the differential behavior which can be completely described by one analytic function $\kappa \equiv \partial_z\bar{\omega}$. The linearity in z also makes it easy to find an analytic equation where the lens equation is embedded. Using $z-\omega = f(\bar{z};\bar{x}_j)$ and $\bar{z}-\bar{\omega} = f(z;x_j)$,

$$z-\omega = f\left(f(z;x_j)+\bar{\omega};\bar{x}_j\right) \tag{4}$$

If we let $H \equiv z_1 z_2 z_3$ and $G \equiv \epsilon_1 z_2 z_3 + \epsilon_2 z_3 z_1 + \epsilon_3 z_1 z_2 = \sum_{\rm cyc} \epsilon_i z_j z_k$, then $f = G/H$, and it is simple to see that equation (4) is a tenth order polynomial equation. If we let $\bar\omega_j \equiv \bar\omega - \bar x_j$,

$$0 = (z-\omega)(G+\bar\omega_1 H)(G+\bar\omega_2 H)(G+\bar\omega_3 H) - \sum_{\rm cyc} H\epsilon_i (G+\bar\omega_j H)(G+\bar\omega_k H) \tag{5}$$

An analytic polynomial equation has the same number of solutions as the order (see any textbook on complex variable or mathematical physics), there can be up to ten images in a triple lensing. There are only four images for $\omega = \infty$, the number of images changes by two at a caustic crossing, and the caustic curves form heierachical structures of domains for high multiplicity images. There are triple lenses with domain $\mathcal{D}^3$ the sources therein produce ten images, the maximum possible number of images (Rhie 1997). As we repeatedly emphasized in Rhie (1997), the triple lens equation is equivalent to the tenth order polynomial analytic equation only in the domain $\mathcal{D}^3$, and the statement on the equivalence in section 3 of Gaudi, Naber, and Sackett (1998) seems to be a misquote which can mislead the readers to think that the two equations are equivalent everywhere.

In order to fit a triple microlensing light curve, we need to solve the tenth order analytic equation (and select the image solutions that satisfy the lens equation). This can be done numerically using root finders available in the literature, and we only need to type in the coefficients. The coefficients may appear to be cumbersome as declared in Gaudi, Naber, and Sackett (1998), and it is indeed the case if we, for example, calculate the coeffients using Mathematica. The output of an algebraic computing package is (unnecessarily) messy even for the binary lens equation. Thus, it is useful to group (or not to unfold) the coefficients, which is a natural intermediate process in hand calculations.

In the center of mass system, $\epsilon_j x_j = 0$, H and G are written with four coefficient functions, a, b, c, and d: $H = z^3 + az^2 + bz + c$ and $G = z^2 + az + d$, where $a \equiv -(x_1 + x_2 + x_3)$, $b \equiv x_1x_2 + x_1x_3 + x_2x_3$, $c \equiv -x_1x_2x_3$, and $d \equiv \sum_{\rm cyc} \epsilon_i x_j x_k$. If we let $a_\omega \equiv \bar\omega_1 + \bar\omega_2 + \bar\omega_3$, $b_\omega \equiv \bar\omega_1\bar\omega_2 + \bar\omega_2\bar\omega_3 + \bar\omega_3\bar\omega_1$, $c_\omega \equiv \bar\omega_1\bar\omega_2\bar\omega_3$, and $d_\omega \equiv \sum_{\rm cyc} \epsilon_i \bar\omega_j \bar\omega_k$, then equation (5) becomes

$$0 = G^3(z-\omega) + G^2H((z-\omega)a_\omega + 1) + GH^2((z-\omega)b_\omega + a_\omega - \bar\omega) + H^3((z-\omega)c_\omega + d_\omega) \tag{6}$$

If we let $G^3 \equiv H_{0k} z^k$ $(k \le 6)$, $G^2H \equiv H_{1k} z^k$ $(k \le 7)$, $GH^2 \equiv H_{2k} z^k$ $(k \le 8)$, and $H^3 \equiv H_{3k} z^k$ $(k \le 9)$, the equation becomes

$$0 \ = \ \sum_{k=1}^{10} \ {\rm cff}\ (k)\ z^k \tag{7}$$

where the polynimial coefficients are

- $\mathrm{cff}\ (k) \ = (\ H_{0k-1} + H_{1k-1}\ a_\omega + H_{2k-1}\ b_\omega + H_{3k-1}\ c_\omega) - (H_{0k}\ \omega + H_{1k}\ (\omega a_\omega - 1) + H_{2k}\ (\omega b_\omega + a_\omega - \bar\omega) + H_{3k}\ (\omega c_\omega + b_\omega))$

The coefficients H_{ij} are polynomials of a, b, c, and d where the polynomial coefficients are simple combinatoric integers.

- $H_{39} = 1$; $H_{38} = 3a$; $H_{37} = 3b+3a^2$; $H_{36} = 3c+6ab+a^3$; $H_{35} = 6ac+3b^2+3a^2b$; $H_{34} = 6bc + 3a^2c + 3ab^2$; $H_{33} = 3c^2 + 6abc + b^3$; $H_{32} = 3ac^2 + 3b^2c$; $H_{31} = 3bc^2$; $H_{30} = c^3$.
- $H_{28} = 1$; $H_{27} = 3a$; $H_{26} = d + 2b + 3a^2$; $H_{25} = 2ad + 4ab + a^3 + 2c$; $H_{24} = 2db + da^2 + 4ac+2a^2b+b^2$; $H_{23} = 2dc+2dab+2a^2c+ab^2+2bc$; $H_{22} = 2cad+db^2+2abc+c^2$; $H_{21} = 2bcd + ac^2$; $H_{20} = c^2d$
- $H_{17} = 1$; $H_{16} = 3a$; $H_{15} = 2d + 3a^2 + b$; $H_{14} = 4ad + a^3 + 2ab + c$; $H_{13} = d^2 + 2a^2d + 2bd + ba^2 + 2ac$; $H_{12} = ad^2 + 2abd + 2cd + ca^2$; $H_{11} = bd^2 + 2acd$; $H_{10} = cd^2$
- $H_{06} = 1$; $H_{05} = 3a$; $H_{04} = 3d + 3a^2$; $H_{03} = 6ad + a^3$; $H_{02} = 3d^2 + 3a^2d^2$; $H_{01} = 3ad^2$; $H_{00} = d^3$

2.1. Comments

An interested party may download the source file of this manuscript to avoid tying the coefficients. We encourage to check the coefficients, however. It is a quick exercise once one adopts the poor person's calculation with pencil and paper as shown above; also with the free biocomputer which is harder to hack either internally or externally barring the long term process of brainwashing. We also found it useful to test the symmetric cases in rh97 whose image solutions behavior (for example, the number of images) is known. The critical curve is obtained by solving $\kappa = e^{i2\varphi}$ which is an eighth order polynomial equation. The caustic curve is obtained by applying the lens equation to the critical curve solution. It is fine to use $\varphi = [0, 2\pi)$ as the parameter for equal interval sampling. Let $\delta = 2\pi/N$ for N not too large and observe the intervals (or speeds) of the solutions on the critical curve and the caustic curve. Note especially the density of the solutions around the cusps of the caustic curve. It is worth pausing for a moment counting the relative numbers of the solutions on the stellar caustic and planetary caustics for planet systems lenses.

For the measurements and theoretical calculations of the anomalous magnetic moment of the electron, we have consulted Peskin and Schroeder (1999) and Kinoshita (1990). We have considered drawing all the Feynman diagrams to lay out the degree of lengthy squiggly messiness but given up, and our misadventure may be considered an indirect testimony of the degree of mud wrestling the exquisite anomalous magnetic moment requires.

Barring the notion that lensing community may have been chosen to be dealt with laxed scrutiny of nature, we have no doubt that we have no luxury to complain about a bit of algebra we encounter in lensing. In fact, we find it a cherished treasure that low multiplicity point lenses are exactly solvable and their light curves can be reconstructed and interpreted without ambiguities. Microlensing events do share the transiency with the scattering events in accelerator particle physics. The both need high resolution data for minute rare prized signals and complete data to interpret them. The both rely on methodical analyses building from the simpler and dominant events to more rare events. Thus, it is important to have a homogeneous and comprehensive data set of microlensing events where consistencies can be tested within in order to find microlensing planets. The best bet for such data set is microlensing from space. Accelerator particle physics of the last century is at the foundation of the Standard Model or the Theory of Matter. We expect that space microlensing planet search will lay a foundation of extrasolar planet physics within a few years of operation of a small space telescope (Bennett and Rhie 2000). In the comprehensive data set, the so-called high magnification events with only stellar caustic signals will form an independent data set that can be used for a consistency check of the interpretation of the planetary light curves as a subset.

This note is based on the work with D. Bennett for Bennett et al. (1999).

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